ANALYTICAL SOLUTION OF SOLIDIFICATION PROBLEMS

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An analytic solution (in the form of an infinite series) has been obtained for the inverse problem of solidification in the most general three-dimensional case. A numerical example is given.

In the three-dimensional case the temperature distribution in a solidifying crust is described by the Fourier equation

$$\frac{\partial \theta}{\partial \mathrm{Fo}} = \frac{\partial^2 \theta}{\partial \eta_1^2} + \frac{\partial^2 \theta}{\partial \eta_2^2} + \frac{\partial^2 \theta}{\partial \eta_3^2}. \tag{1}$$

The contour of the solidification front is assumed given at any instant of time:

$$\Gamma(\eta_1, \eta_2, \eta_3, \mathrm{Fo}) = 0. \tag{2}$$

The surface contour of the body is given by the equation

$$\Gamma(\eta_1, \eta_2, \eta_3, 0) = 0. \tag{3}$$

The function Γ and its derivatives are assumed continuous.

As usual, the boundary conditions are specified only on the contour of the solidification front:

$$\theta|_{\Gamma=0} = \theta_c, \tag{4}$$

$$-\left.\frac{\partial \theta}{\partial n}\right|_{\Gamma=0} = \left.\frac{dn}{d\,\mathrm{Fo}} + q_1\right|_{\Gamma=0}. \tag{5}$$

The boundary conditions formulated are a natural extension to the three-dimensional case of the formulation of the one-dimensional solidification problem [1].

Condition (5) can be transformed by the method proposed by Ivantsov [2].

For this purpose we consider that in the orthogonal coordinate system n, s, σ the total differential of the dimensionless temperature $d\theta$ is written in the form

$$d\theta = \frac{\partial \theta}{\partial n} dn + \frac{\partial \theta}{\partial s} ds + \frac{\partial \theta}{\partial \sigma} d\sigma + \left(\frac{\partial \theta}{\partial \mathrm{Fo}}\right)_{n,s,\sigma} d\,\mathrm{Fo}. \tag{6}$$

Taking into account the constancy of the temperature on the contour of the solidification front $\Gamma = 0$, we have

$$\left.\left(\frac{\partial \theta}{\partial \mathrm{Fo}}\right)_{n,s,\sigma}\right|_{\Gamma=0} = 0. \tag{7}$$

Moreover, there is, of course, no variation of the temperature along the surface of the solidification front; hence

$$\left.\frac{\partial \theta}{\partial s}\right|_{\Gamma=0} = 0; \quad \left.\frac{\partial \theta}{\partial \sigma}\right|_{\Gamma=0} = 0. \tag{8}$$

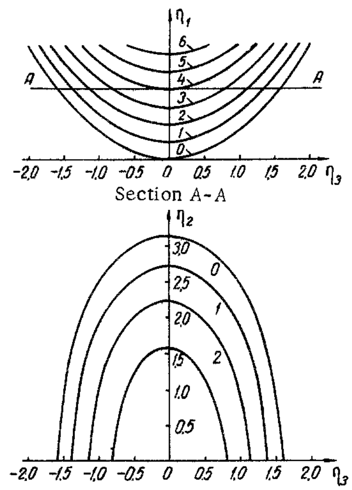

Fig. 1. Contours of body and isosolidus for successive stages of solidification of an ingot of elliptical cross section with a paraboloidal bottom: 0) outer contour of body (Fo = 0); 1) Fo = 0.25; 2) 0.5; 3) 0.75; 4) 1.0; 5) 1.25; 6) 1.5.

Thence from expression (6)

$$\left.\frac{dn}{d\,\mathrm{Fo}}\right|_{\Gamma=0} = \frac{d\theta}{d\,\mathrm{Fo}} \Big/ \left.\frac{\partial \theta}{\partial n}\right|_{\Gamma=0}. \tag{9}$$

As usual, the expression for $\partial\theta/\partial n$ can be written in η_1, η_2, η_3 coordinates in the form

$$\frac{\partial \theta}{\partial n} = \sqrt{\left(\frac{\partial \theta}{\partial \eta_1}\right)^2 + \left(\frac{\partial \theta}{\partial \eta_2}\right)^2 + \left(\frac{\partial \theta}{\partial \eta_3}\right)^2}. \tag{10}$$

Then condition (5) becomes

$$-\left.\left[\left(\frac{\partial \theta}{\partial \eta_1}\right)^2 + \left(\frac{\partial \theta}{\partial \eta_2}\right)^2 + \left(\frac{\partial \theta}{\partial \eta_3}\right)^2\right]\right|_{\Gamma=0} = \left.\frac{d\theta}{d\,\mathrm{Fo}}\right|_{\Gamma=0} + q_l \left.\sqrt{\left(\frac{\partial \theta}{\partial \eta_1}\right)^2 + \left(\frac{\partial \theta}{\partial \eta_2}\right)^2 + \left(\frac{\partial \theta}{\partial \eta_3}\right)^2}\right|_{\Gamma=0}. \tag{11}$$

We find the solution of Eq. (1) with boundary conditions (4) and (11) and given contour Γ in the form of a series that is a natural generalization of the one-dimensional solution [1]:

$$\theta = \sum_{l=0}^{\infty} \frac{A_l(\eta_1, \eta_2, \eta_3, \mathrm{Fo})}{l!} \Gamma^l. \tag{12}$$

Then

$$\frac{\partial \theta}{\partial \eta_i} = \sum_{l=0}^{\infty} \frac{\frac{\partial \Gamma}{\partial \eta_i} A_{l+1} + \frac{\partial}{\partial \eta_i} A_l}{l!} \Gamma^l, \tag{13}$$

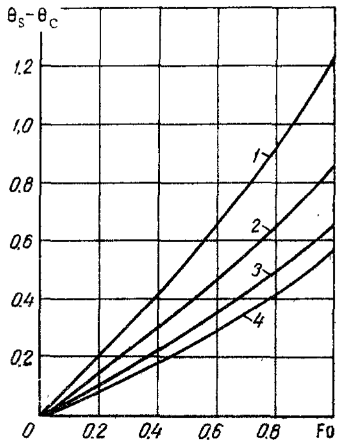

Fig. 2. Variation with time of the difference $\Theta_c - \Theta_s$ at the surface of an ingot of elliptical cross section with a paraboloidal bottom. The curves were constructed for surface points with the coordinates: 1) (0, 0, 0); 2) (1, 3.16, 0); 3) (1, 2, 1.225); 4) (1, 0, 1.58).

$$\frac{\partial^2 \theta}{\partial \eta_i^2} = \sum_{l=0}^{\infty} \left[\frac{\partial^2 \Gamma}{\partial \eta_i^2} A_{l+1} + 2 \frac{\partial \Gamma}{\partial \eta_i} \frac{\partial A_{l+1}}{\partial \eta_i} + \frac{\partial^2 A_l}{\partial \eta_i^2} + \left(\frac{\partial \Gamma}{\partial \eta_i}\right)^2 A_{l+2} \right] \Gamma^l / l! \tag{14}$$

$$\frac{d\theta}{d\,\mathrm{Fo}} = \left(\frac{\partial \theta}{\partial\,\mathrm{Fo}}\right)_{\eta_i} = \sum_{l=0}^{\infty} \frac{\left(\frac{\partial \Gamma}{\partial\,\mathrm{Fo}}\right)_{\eta_i} A_{l+1} + \left(\frac{\partial}{\partial\,\mathrm{Fo}} A_l\right)_{\eta_i}}{l!} \Gamma^l. \tag{15}$$

Substituting expressions (13)–(15) into Eq. (1) and equating the coefficients of like powers of Γ to zero, we obtain the following recurrence relation:

$$A_{l+2} = \frac{1}{\left(\frac{\partial \Gamma}{\partial \eta_1}\right)^2 + \left(\frac{\partial \Gamma}{\partial \eta_2}\right)^2 + \left(\frac{\partial \Gamma}{\partial \eta_3}\right)^2} \left\{ \left(\frac{\partial}{\partial\,\mathrm{Fo}} A_l\right)_{\eta_i} - \frac{\partial^2 A_l}{\partial \eta_1^2} - \frac{\partial^2 A_l}{\partial \eta_2^2} - \frac{\partial^2 A_l}{\partial \eta_3^2} - \left[\frac{\partial^2 \Gamma}{\partial \eta_1^2} + \frac{\partial^2 \Gamma}{\partial \eta_2^2} + \frac{\partial^2 \Gamma}{\partial \eta_3^2} - \left(\frac{\partial \Gamma}{\partial\,\mathrm{Fo}}\right)_{\eta_i} \right] A_{l+1} - 2 \left(\frac{\partial \Gamma}{\partial \eta_1} \frac{\partial A_{l+1}}{\partial \eta_1} + \frac{\partial \Gamma}{\partial \eta_2} \frac{\partial A_{l+1}}{\partial \eta_2} + \frac{\partial \Gamma}{\partial \eta_3} \frac{\partial A_{l+1}}{\partial \eta_3} \right) \right\}. \tag{16}$$

The coefficients A_0 and A_1 are determined from the boundary conditions (4) and (11), respectively:

$$A_0 = \theta_c, \tag{17}$$

$$A_1 = - \left. \frac{\left(\frac{\partial \Gamma}{\partial\,\mathrm{Fo}}\right)_{\eta_i} + q_1 \sqrt{\left(\frac{\partial \Gamma}{\partial \eta_1}\right)^2 + \left(\frac{\partial \Gamma}{\partial \eta_2}\right)^2 + \left(\frac{\partial \Gamma}{\partial \eta_3}\right)^2}}{\left(\frac{\partial \Gamma}{\partial \eta_1}\right)^2 + \left(\frac{\partial \Gamma}{\partial \eta_2}\right)^2 + \left(\frac{\partial \Gamma}{\partial \eta_3}\right)^2} \right|_{\Gamma=0}. \tag{18}$$

In general, it is difficult to prove the convergence of series of (13) at values of the coefficients A_l determined by expressions (16)–(18). However, the expressions presented give a solution of the problem at least for certain cases of practical importance.

As an example we have considered the solidification of an ingot of elliptical cross section with a paraboloidal bottom.

In this case the function Γ is given in the form

$$\Gamma = \eta_1 - \alpha_2 \eta_2^2 - \alpha_3 \eta_3^2 - k\,\mathrm{Fo};$$
$$q_1 = 0. \tag{19}$$

The contours of the bottom of this ingot and the contours of the isosolidus for successive stages of solidification are shown in Fig. 1.

In the calculation it was assumed that $\alpha_2 = 0.1$; $\alpha_3 = 0.4$; $k = 1$. Confining ourselves to the first four terms of series (12), using expressions (18) and (16) to calculate the coefficients A_l, and considering that at the surface of the body $\Gamma|_s = k\mathrm{Fo}$, we obtain

$$\theta_s \simeq \theta_c - \frac{\mathrm{Fo}}{1 + 4(0.01\eta_2^2 + 0.16\eta_3^2)} - \frac{16(0.001\eta_2^2 + 0.064\eta_3^2)\,\mathrm{Fo}^2}{[1 + 4(0.01\eta_2^2 + 0.16\eta_3^2)]^3} - \left\{ \frac{0.226}{[1 + 4(0.01\eta_2^2 + 0.16\eta_3^2)]^3} - \frac{64(0.0001\eta_2^2 + 0.0256\eta_3^2)}{[1 + 4(0.01\eta_2^2 + 0.16\eta_3^2)]^4} + \frac{512(0.001\eta_2^2 + 0.064\eta_3^2)^2}{[1 + 4(0.01\eta_2^2 + 0.16\eta_3^2)]^5} \right\} \mathrm{Fo}^3.$$

In order to calculate the dimensionless temperature θ_s we took points in the bottom part of the ingot surface with coordinates (0, 0, 0), (1, 3.16, 0), (1, 2, 1.225).

The variation of θ_s with time at each of these points, respectively, is given by the expressions

$$\theta_s \simeq \theta_c - 1.000\,\mathrm{Fo} - 0.226\,\mathrm{Fo}^3,$$
$$\theta_s \simeq \theta_c - 0.715\,\mathrm{Fo} - 0.058\,\mathrm{Fo}^2 - 0.074\,\mathrm{Fo}^3,$$
$$\theta_s \simeq \theta_c - 0.472\,\mathrm{Fo} - 0.168\,\mathrm{Fo}^2 - 0.0200\,\mathrm{Fo}^3,$$
$$\theta_s \simeq \theta_c - 0.385\,\mathrm{Fo} - 0.145\,\mathrm{Fo}^2 - 0.0340\,\mathrm{Fo}^3.$$

These can be used at least for $\mathrm{Fo} \le 1$; at larger values of Fo a larger number of terms of series (12) must be determined.

Figure 2 shows the variation with time of the difference $\theta_c - \theta_s$ calculated from the formulas presented.

NOTATION

$\theta = ct/r$ is dimensionless temperature; $\theta_c = ct_c/r$; $\theta_s = ct_s/r$; t is temperature; t_c is crystallization temperature; t_s is surface temperature; c is specific heat; r is specific latent heat of crystallization; $Fo = a\tau/\tilde{X}^2$ is dimensionless time; a is thermal diffusivity; τ is time; $\tilde{X}$ is the characteristic dimension of body; $\eta_i = x_i/\tilde{X}$ is the dimensionless coordinate; n is the dimensionless normal; $q_l = Q_l\tilde{X}c/\lambda r$ is dimensionless heat flow from liquid core of ingot to crust; Q_l is the corresponding dimensional heat flow; λ is thermal conductivity.

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